# Modelling with Differential Equations

## Questions

**Q1.**

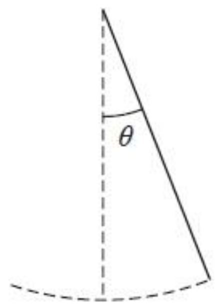


**Figure 3**

The motion of a pendulum, shown in Figure 3, is modelled by the differential equation

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

where *θ* is the angle, in radians, that the pendulum makes with the downward vertical, *t* seconds after it begins to move.

(a) (i) Show that a particular solution of the differential equation is

$$\theta = \frac{1}{12}t\sin 3t$$

**(4)**

(ii) Hence, find the general solution of the differential equation.

**(4)**

Initially, the pendulum

- makes an angle of $\frac{\pi}{3}$ radians with the downward vertical
- is at rest

Given that, 10 seconds after it begins to move, the pendulum makes an angle of *α* radians with the downward vertical,

(b) determine, according to the model, the value of *α* to 3 significant figures.

**(4)**

Given that the true value of *α* is 0.62

(c) evaluate the model.

**(1)**

The differential equation

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

models the motion of the pendulum as moving with forced harmonic motion.

(d) Refine the differential equation so that the motion of the pendulum is simple harmonic motion.

**(1)**

**(Total for question = 14 marks)**

**Q2.**

An engineer is investigating the motion of a sprung diving board at a swimming pool.
Let *E* be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.
A diver jumps from the diving board.
The vertical displacement, *h* cm, of the end of the diving board above *E* is modelled by the differential equation

$$4\frac{d^2h}{dt^2} + 4\frac{dh}{dt} + 37h = 0$$

where *t* seconds is the time after the diver jumps.

(a) Find a general solution of the differential equation.

**(2)**

When $t = 0$, the end of the diving board is 20 cm below *E* and is moving upwards with a speed of 55 cm $s^{-1}$.

(b) Find, according to the model, the maximum vertical displacement of the end of the diving board above *E*.

**(8)**

(c) Comment on the suitability of the model for large values of *t*.

**(2)**

**(Total for question = 12 marks)**

**Q3.**

A scientist is studying the effect of introducing a population of white-clawed crayfish into a population of signal crayfish.
At time $t$ years, the number of white-clawed crayfish, $w$, and the number of signal crayfish, $s$, are modelled by the differential equations

$$\frac{dw}{dt} = \frac{5}{2}(w - s)$$
$$\frac{ds}{dt} = \frac{2}{5}w - 90e^{-t}$$

(a) Show that

$$2\frac{d^2w}{dt^2} - 5\frac{dw}{dt} + 2w = 450e^{-t}$$

**(3)**

(b) Find a general solution for the number of white-clawed crayfish at time $t$ years.

**(6)**

(c) Find a general solution for the number of signal crayfish at time $t$ years.

**(2)**

The model predicts that, at time $T$ years, the population of white-clawed crayfish will have died out.

Given that $w = 65$ and $s = 85$ when $t = 0$

(d) find the value of $T$, giving your answer to 3 decimal places.

**(6)**

(e) Suggest a limitation of the model.

**(1)**

**(Total for question = 18 marks)**

**Q4.**

At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time $t$ years after the survey began, the number of foxes, $f$, and the number of rabbits, $r$, on the island are modelled by the differential equations

$$\frac{df}{dt} = 0.2f + 0.1r$$
$$\frac{dr}{dt} = -0.2f + 0.4r$$

(a) Show that $\frac{d^2f}{dt^2} - 0.6\frac{df}{dt} + 0.1f = 0$

**(3)**

(b) Find a general solution for the number of foxes on the island at time $t$ years.

**(4)**

(c) Hence find a general solution for the number of rabbits on the island at time $t$ years.

**(3)**

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

**(7)**

**(Total for question = 17 marks)**

**Q5.**

A sample of bacteria in a sealed container is being studied.

The number of bacteria, $P$, in thousands, is modelled by the differential equation

$$(1+t)\frac{dP}{dt} + P = t^{\frac{1}{2}}(1+t)$$

where $t$ is the time in hours after the start of the study.

Initially, there are exactly 5000 bacteria in the container.

(a) Determine, according to the model, the number of bacteria in the container 8 hours after the start of the study.

**(6)**

(b) Find, according to the model, the rate of change of the number of bacteria in the container 4 hours after the start of the study.

**(4)**

(c) State a limitation of the model.

**(1)**

**(Total for question = 11 marks)**

**Q6.**

A pond initially contains 1000 litres of unpolluted water.

The pond is leaking at a constant rate of 20 litres per day.

It is suspected that contaminated water flows into the pond at a constant rate of 25 litres per day and that the contaminated water contains 2 grams of pollutant in every litre of water.

It is assumed that the pollutant instantly dissolves throughout the pond upon entry.

Given that there are $x$ grams of the pollutant in the pond after $t$ days,

(a) show that the situation can be modelled by the differential equation,

$$\frac{dx}{dt} = 50 - \frac{4x}{200+t}$$

**(4)**

(b) Hence find the number of grams of pollutant in the pond after 8 days.

**(5)**

(c) Explain how the model could be refined.

**(1)**

**(Total for question = 10 marks)**

**Q7.**

A tank at a chemical plant has a capacity of 250 litres. The tank initially contains 100 litres of pure water.

Salt water enters the tank at a rate of 3 litres every minute. Each litre of salt water entering the tank contains 1 gram of salt.

It is assumed that the salt water mixes instantly with the contents of the tank upon entry.

At the instant when the salt water begins to enter the tank, a valve is opened at the bottom of the tank and the solution in the tank flows out at a rate of 2 litres per minute.

Given that there are *S* grams of salt in the tank after *t* minutes,

(a) show that the situation can be modelled by the differential equation

$$\frac{dS}{dt} = 3 - \frac{2S}{100 + t}$$

**(4)**

(b) Hence find the number of grams of salt in the tank after 10 minutes.

**(5)**

When the concentration of salt in the tank reaches 0.9 grams per litre, the valve at the bottom of the tank must be closed.

(c) Find, to the nearest minute, when the valve would need to be closed.

**(3)**

(d) Evaluate the model.

**(1)**

**(Total for question = 13 marks)**

**Q8.**

A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.
The vertical displacement, $x$ metres, of the top of the capsule below its initial position at time $t$ seconds is modelled by the differential equation,

$$m\frac{d^2x}{dt^2}+4\frac{dx}{dt}+x=200\cos t, \quad t\geqslant 0$$

where $m$ is the mass of the capsule including its passengers, in thousands of kilograms.

The maximum permissible weight for the capsule, including its passengers, is 30 000N.

Taking the value of $g$ to be 10 $ms^{-2}$ and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of $m$ is 3

(ii) show that a particular solution to the differential equation is

$$x = 40\sin t - 20\cos t$$

(iii) hence find the general solution of the differential equation.

**(8)**

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

**(4)**

**(Total for question = 12 marks)**

**Q9.**

A scientist is investigating the concentration of antibodies in the bloodstream of a patient following a vaccination.

The concentration of antibodies, $x$, measured in micrograms (μg) per millilitre (ml) of blood, is modelled by the differential equation

$$100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 26$$

where $t$ is the number of weeks since the vaccination was given.

(a) Find a general solution of the differential equation.

**(4)**

Initially,

- there are no antibodies in the bloodstream of the patient
- the concentration of antibodies is estimated to be increasing at 10 μg/ml per week

(b) Find, according to the model, the maximum concentration of antibodies in the bloodstream of the patient after the vaccination.

**(8)**

A second dose of the vaccine has to be given to try to ensure that it is fully effective.
It is only safe to give the second dose if the concentration of antibodies in the bloodstream of the patient is less than 5 μg/ml.

(c) Determine whether, according to the model, it is safe to give the second dose of the vaccine to the patient exactly 10 weeks after the first dose.

**(2)**

**(Total for question = 14 marks)**

**Q10.**

Two compounds, *X* and *Y*, are involved in a chemical reaction. The amounts in grams of these compounds, *t* minutes after the reaction starts, are *x* and *y* respectively and are modelled by the differential equations

$$\frac{dx}{dt} = -5x + 10y - 30$$

$$\frac{dy}{dt} = -2x + 3y - 4$$

(a) Show that

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 50$$

**(3)**

(b) Find, according to the model, a general solution for the amount in grams of compound *X* present at time *t* minutes.

**(6)**

(c) Find, according to the model, a general solution for the amount in grams of compound *Y* present at time *t* minutes.

**(3)**

Given that $x = 2$ and $y = 5$ when $t = 0$

(d) find

(i) the particular solution for *x*,
(ii) the particular solution for *y*.

**(4)**

A scientist thinks that the chemical reaction will have stopped after 8 minutes.

(e) Explain whether this is supported by the model.

**(1)**

**(Total for question = 17 marks)**

## Mark Scheme – Modelling wit Differential Equations

**Q1.**

| Question | Scheme | | Marks | AOs |
|---|---|---|---|---|
| (a)(i) | $\frac{d\theta}{dt} = \alpha \sin 3t + \beta t \cos 3t$ and $\frac{d^2\theta}{dt^2} = \delta \cos 3t + \gamma t \sin 3t$ | Let $\theta = \lambda t \sin 3t$ $\frac{d\theta}{dt} = \alpha \sin 3t + \beta t \cos 3t$ and $\frac{d^2\theta}{dt^2} = \delta \cos 3t + \gamma t \sin 3t$ | M1 | 1.1b |
| | $\frac{d\theta}{dt} = \frac{1}{12}\sin 3t + \frac{1}{4}t \cos 3t$ and $\frac{d^2\theta}{dt^2} = \frac{1}{4}\cos 3t + \frac{1}{4}\cos 3t - \frac{3}{4}t \sin 3t$ $= \frac{1}{2}\cos 3t - \frac{3}{4}t \sin 3t$ | $\frac{d\theta}{dt} = \lambda \sin 3t + 3\lambda t \cos 3t$ and $\frac{d^2\theta}{dt^2} = 3\lambda \cos 3t + 3\lambda \cos 3t - 9\lambda t \sin 3t$ $= 6\lambda \cos 3t - 9\lambda t \sin 3t$ | A1 | 1.1b |
| | $\frac{1}{2}\cos 3t - \frac{3}{4}t \sin 3t + 9\left(\frac{1}{12}t \sin 3t\right) =\ldots$ | $6\lambda \cos 3t - 9\lambda t \sin 3t + 9(\lambda t \sin 3t) = \frac{1}{2}\cos 3t \Rightarrow \lambda =\ldots$ | dM1 | 3.4 |
| | $= \frac{1}{2}\cos 3t$ so PI is $\theta = \frac{1}{12}t \sin 3t$ * | $\theta = \frac{1}{12}t \sin 3t$ * | A1* | 2.1 |
| | | | (4) | |
| (a)(ii) | $m^2 + 9 = 0 \Rightarrow m = \pm 3i$ | | M1 | 1.1b |
| | $\theta = A \cos 3t + B \sin 3t$ | | A1 | 1.1b |
| | $(\theta =) CF + PI$ | | dM1 | 1.1b |
| | $\theta = A \cos 3t + B \sin 3t + \frac{1}{12}t \sin 3t$ | | A1 | 1.1b |
| | | | (4) | |
| (b) | $t = 0,\ \theta = \frac{\pi}{3} \Rightarrow A = \ldots\left\{\frac{\pi}{3}\right\}$ | | M1 | 3.4 |
| | $t = 0,\ \frac{d\theta}{dt} = -3A \sin 3t + 3B \cos 3t + \frac{1}{12}\sin 3t + \frac{1}{4}t \cos 3t = 0 \Rightarrow B = \ldots\{0\}$ | | M1 | 3.4 |
| | $\alpha = \frac{\pi}{3}\cos(3 \times 10) + \frac{1}{12}(10)\sin(3 \times 10) =\ldots$ | | ddM1 | 1.1b |
| | $\alpha = \pm$awrt 0.662 | | A1 | 3.4 |
| | | | (4) | |
| (c) | 0.662 is close to 0.62 so a good model (at $t = 10$) | | B1ft | 3.5a |
| | | | (1) | |
| (d) | $\frac{d^2\theta}{dt^2} + 9\theta = 0$ oe | | B1 | 3.5c |
| | | | (1) | |

| (14 marks) |
|---|
| **Notes:** |
| **(a)(i) Note: mark (a) as a whole**<br>**M1:** Differentiates the given PI twice using the product rule to achieve the required form. Alternatively, uses a correct form for the PI and differentiates twice using the product rule to achieve the required form. A correct form may involve other terms with coefficients that will be zero, e.g. $\theta = \lambda t \sin 3t + \mu t \cos 3t$ is fine. Also allow e.g $\theta = \lambda t \sin \quad \omega t$<br>**A1:** Correct derivatives.<br>**dM1:** Depends on first M, substitutes into the given differential equation and attempts to simplify. In the Alt they must go on to find value for $\lambda$.<br>**A1*:** Achieves $\frac{1}{2}\cos 3t$and makes a minimal conclusion (e.g //). Alternatively reaches the correct PI. |
| **(a)(ii)**<br>**M1:** Uses the model to form and solve the auxiliary equation. Accept $m^2 + 9 = 0 \rightarrow m = \pm 3i$ or $\pm$ 3<br>**A1**: Correct complementary function. Must be in terms of *t* but allow recovery if initially in terms of *x* but changed later.<br>**dM1:** Dependent on the previous method mark. Finds the general solution by adding the particular integral to the complementary function.<br>**A1:** Correct general solution including "$\theta =$ ", which may be recovered in part (b). |
| **(b)**<br>**M1:** Uses the initial conditions of the model, $t = 0,\ \theta = \frac{\pi}{3}$to find a value for a constant.<br>**M1:** Differentiates the general solution and uses the initial conditions of the model $t = 0,\ \frac{d\theta}{dt} = 0$ to find a value for the other constant.<br>**ddM1:** Dependent on both previous method marks. Substitutes $t = 10$ into their particular solution. If not substitution is seen, accept any value as the attempt as long as they have found all relevant constants.<br>**A1:** Accept awrt ±0.662 |
| **(c)**<br>**B1ft:** Makes a quantitative comparison of the size of their answer to part (b) with 0.62 and makes conclusion (e.g. good model). Follow through on their answer to (b) and draws an appropriate conclusion about the model. Accept "not reasonable" as long as it is supported with evidence but there must be some instructive comparison and a conclusion about the model - not just stating how much it is out. The reason given must be correct.<br>Accept e.g. a correct percentage error with reasonable conclusion, or statement approximately equal with conclusion.<br>Do not accept e.g. "does not agree to 1 s.f." or "out by 0.6" as these lacks context. Do not accept arguments based solely on a difference in sign, they must be referring to the relative size of angle. |
| **(d)**<br>**B1:** Refines the model, accept any constant on the right hand side. |

**Q2.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $4m^2+4m+37=0 \Rightarrow m=-\frac{1}{2}\pm 3\text{i}$ | M1 | 1.1b |
| | $h=\text{e}^{-0.5t}\left(A\cos 3t+B\sin 3t\right)$ | A1 | 1.1b |
| | | (2) | |
| (b) | $t=0,\ h=-20\Rightarrow A=-20$ | M1 | 3.4 |
| | $\frac{\text{d}h}{\text{d}t}=-0.5\text{e}^{-0.5t}\left(A\cos 3t+B\sin 3t\right)+\text{e}^{-0.5t}\left(-3A\sin 3t+3B\cos 3t\right)$ <br> $t=0,\ \frac{\text{d}h}{\text{d}t}=55\Rightarrow B=\ldots$ (NB $B=15$) | M1 | 3.4 |
| | $(h=)\text{e}^{-0.5t}(15\sin 3t-20\cos 3t)$ | A1 | 1.1b |
| | $-0.5\text{e}^{-0.5t}(15\sin 3t-20\cos 3t)+\text{e}^{-0.5t}(60\sin 3t+45\cos 3t)=0$ <br> or e.g. <br> $-0.5\text{e}^{-0.5t}(15\sin 3t-20\cos 3t)+\frac{25\sqrt{37}}{2}\text{e}^{-0.5t}\sin\left(3t+\arctan\frac{22}{21}\right)=0$ <br> $\Rightarrow t=\ldots$ | M1 | 3.1b |
| | $\tan 3t=-\frac{22}{21}$ or e.g. $3t+\tan^{-1}\frac{22}{21}=0$ | A1 **M1 on ePEN** | 2.1 |
| | $t=0.778$ s | A1 | 1.1b |
| | $h=\text{e}^{-0.5\times\text{"}0.778\text{"}}\left(15\sin\left(3\times\text{"}0.778\text{"}\right)-20\cos\left(3\times\text{"}0.778\text{"}\right)\right)$ | dM1 | 1.1b |
| | $=16.7$ cm | A1 | 3.2a |
| | | (8) | |
| (c) | E.g. considers large values of $t$ in the model for $h$ **or** states that for large values of $t$, $h$ becomes smaller or becomes zero | M1 | 3.4 |
| | E.g. <br> • The value of $h$ is very small when $t$ is large and this is likely to be correct (as the displacement of end of the board should get smaller and smaller) <br> • This suggests the model is suitable <br> • This is realistic <br> • This is suitable as the board will tend towards its equilibrium position <br> • When $t$ is large the value of $h$ is never zero so the model is not really appropriate for large values of $t$ | A1 **B1 on ePEN** | 3.2b |
| | | (2) | |
| | | **(12 marks)** | |

| Notes |
|---|
| (a)<br>M1: Uses the model to form and solve the auxiliary equation $4m^2+4m+37=0$<br>See General Guidance for awarding this mark. This can be implied by correct values for $m$ (from calculator)<br>A1: Correct general solution including "$h=$"<br>(b)<br>M1: Uses the model and the initial conditions to establish the value of "$A$". Need to see $t=0$ and $h=\pm 20$ leading to a value for "$A$". This may be implied by $A=-20$ or $A=20$.<br>M1: Differentiates their model using the product rule and uses the initial conditions, $t=0$ with $\mathrm{d}h/\mathrm{d}t=\pm 55$, to establish the value of "$B$"<br>A1: Correct particular solution or correct values for $A$ and $B$<br>M1: Uses their solution to the model with a correct strategy to obtain a value for $t$ e.g. differentiates or uses their derivative from earlier, sets equal to zero and solves for $t$<br>A1(M1 on ePEN): Correct equation for $t$<br>A1: Correct value for $t$ (allow awrt 0.778 if necessary) but this value may be implied.<br>**d**M1: Uses the model and their **positive** value for $t$ to find the maximum displacement - **if their $t$ is incorrect there must be some indication that they are using their $h$ and not just a number written down. E.g. must see substitution into their $h$ or they re-state their $h$ and obtain a value for $h$.**<br>**Dependent on all the previous method marks**<br>A1: Correct value (awrt 16.7 (units not needed))<br>(c)<br>M1: Considers the model for large values of $t$ either by substituting values or by considering the expression and commenting on its behaviour for large values of $t$. E.g. as $t\to\infty$, $h\to 0$ or as $t\to\infty$, $e^{-0.5t}\to 0$ or as $t\to\infty$ the oscillations become smaller etc.<br>A1: Makes a suitable comment – see scheme for examples |

**Q3.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $\frac{d^2w}{dt^2}=\frac{5}{2}\left(\frac{dw}{dt}-\frac{ds}{dt}\right)$ or $\frac{ds}{dt}=\frac{dw}{dt}-\frac{2}{5}\frac{d^2w}{dt^2}$ o.e. | B1 | 1.1b |
| | $\frac{ds}{dt}=\frac{dw}{dt}-\frac{2}{5}\frac{d^2w}{dt^2}\Rightarrow\frac{dw}{dt}-\frac{2}{5}\frac{d^2w}{dt^2}=\frac{2}{5}w-90e^{-t}$ | M1 | 2.1 |
| | $2\frac{d^2w}{dt^2}-5\frac{dw}{dt}+2w=450e^{-t}$ * | A1* | 1.1b |
| | | (3) | |
| (b) | $2m^2-5m+2=0\Rightarrow m=...$ | M1 | 3.4 |
| | $m=2,\frac{1}{2}$ | A1 | 1.1b |
| | $(w)=Ae^{\alpha t}+Be^{\beta t}$ | M1 | 3.4 |
| | $(w)=Ae^{0.5t}+Be^{2t}$ | A1 | 1.1b |
| | PI: Try $w=ke^{-t}\Rightarrow\frac{dw}{dt}=-ke^{-t}\Rightarrow\frac{d^2w}{dt^2}=ke^{-t}$<br>$2ke^{-t}+5ke^{-t}+2ke^{-t}=450e^{-t}\Rightarrow k=...$ | M1 | 3.4 |
| | $w=$ 'their C.F.' $+\ 50e^{-t}$<br>$\left(w=Ae^{0.5t}+Be^{2t}+50e^{-t}\right)$ | A1ft | 1.1b |
| | | (6) | |
| (c) | $s=w-\frac{2}{5}\frac{dw}{dt}=Ae^{0.5t}+Be^{2t}+50e^{-t}-\frac{2}{5}\left(\frac{A}{2}e^{0.5t}+2Be^{2t}-50e^{-t}\right)$ | M1 | 3.4 |
| | $s=\frac{4A}{5}e^{0.5t}+\frac{B}{5}e^{2t}+70e^{-t}$ | A1 | 1.1b |
| | | (2) | |
| (d) | $65=A+B+50,\ 85=\frac{4A}{5}+\frac{B}{5}+70\Rightarrow A=...,B=...$<br>$(\text{NB } A=20\ \ B=-5)$ | M1 | 3.3 |
| | $w=0\Rightarrow 20e^{0.5t}-5e^{2t}+50e^{-t}=0$ | dM1 | 1.1b |
| | $e^{3t}-4e^{1.5t}-10(=0)$ or a multiple | A1 | 3.1a |
| | $e^{1.5t}=\frac{4\pm\sqrt{4^2-4\times(1)(-10)}}{2}$ | M1 | 1.1b |
| | $1.5t=\ln\left(\frac{4+\sqrt{56}}{2}\right)$ | M1 | 2.3 |
| | $T=\frac{2}{3}\ln\left(\frac{4+\sqrt{56}}{2}\right)=$ awrt 1.165 | A1 | 3.2a |
| | | (6) | |
| (e) | E.g.<br>• Either population becomes negative which is not possible<br>• When the white-clawed crayfish have died out, the model will not be valid | B1 | 3.5b |
| | | (1) | |
| | | | (18 marks) |

| Notes |
|---|
| (a)<br>B1: Differentiates the first equation with respect to $t$ correctly.<br>M1: Substitutes $\frac{ds}{dt}$ into their derivative.<br>A1*: Achieves the printed answer with no errors.<br><br>(b) **Note: All the mark except the final A1 are available if they use other variables.**<br>M1: Uses the model to form and solve the Auxiliary Equation.<br>A1: Correct roots of the AE.<br>M1: Uses the model to form the Complementary Function for their roots (they may be complex roots)<br>A1: Correct CF<br>M1: Chooses the correct form of the PI according to the model and uses a complete method to find the PI. Uses $w = ke^{-t}$ finds both $\frac{dw}{dt}$ and $\frac{d^2w}{dt^2}$ substitutes into the differential equation and find the value of $k$.<br>A1ft: Dependent on all three of the previous method marks. Following through on their CF only to give $w =$ 'their CF' $+ 50e^{-t}$<br><br>(c)<br>M1: Substitutes into the first equation the answer for part (b) in place of $w$ and the derivative of their (b) in place of $\frac{dw}{dt}$. If they rearrange to make $S$ the subject first and make a slip but still substitutes for $w$ and $\frac{dw}{dt}$ allow this mark.<br>A1: Correct simplified equation.<br><br>(d)<br>M1: Uses the initial conditions $t = 0, w = 65$ and $s = 85$ to form simulations equations and solves to find the values of their constants<br>dM1: Dependent on the previous method mark. Sets $w = 0$<br>A1: Processes the indices correctly to obtain a 3-term quadratic equation in terms of $e^{1.5t}$. It does not need to all be on one side and condone missing $= 0$.<br>M1: Solves their three-term quadratic (3TQ) to reach $e^{pt} = q$<br>M1: Correct use of logarithms to reach $pt = \ln q$ where $q > 0$ and rejects the other solution<br>A1: awrt 1.165 |

| |
|---|
| **Note:** the final 3 marks only can be implied by a correct answer following the correct 3-term quadratic equation in terms of $e^{1.5t}$<br><br>(e)<br>B1: Suggests a suitable limitation of the model, not valid when negative population<br>Any mention of other factors such as does not take into account e.g. other predictors, fishing, disease, lack of food etc is B0 |

**Q4.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $r=10\frac{df}{dt}-2f \Rightarrow \frac{dr}{dt}=10\frac{d^2f}{dt^2}-2\frac{df}{dt}$ | M1 | 2.1 |
| | $10\frac{d^2f}{dt^2}-2\frac{df}{dt}=-0.2f+0.4\left(10\frac{df}{dt}-2f\right)$ | M1 | 2.1 |
| | $\frac{d^2f}{dt^2}-0.6\frac{df}{dt}+0.1f=0$* | A1* | 1.1b |
| | | (3) | |
| (b) | $m^2-0.6m+0.1=0 \Rightarrow m=\frac{0.6\pm\sqrt{0.6^2-4\times0.1}}{2}$ | M1 | 3.4 |
| | $m=0.3\pm0.1\text{i}$ | A1 | 1.1b |
| | $f=e^{\alpha t}(A\cos\beta t+B\sin\beta t)$ | M1 | 3.4 |
| | $f=e^{0.3t}(A\cos0.1t+B\sin0.1t)$ | A1 | 1.1b |
| | | (4) | |
| (c) | $\frac{df}{dt}=0.3e^{0.3t}(A\cos0.1t+B\sin0.1t)+0.1e^{0.3t}(B\cos0.1t-A\sin0.1t)$ | M1 | 3.4 |
| | $r=10\frac{df}{dt}-2f$ <br> $=e^{0.3t}((3A+B)\cos0.1t+(3B-A)\sin0.1t)-2e^{0.3t}(A\cos0.1t+B\sin0.1t)$ | M1 | 3.4 |
| | $r=e^{0.3t}((A+B)\cos0.1t+(B-A)\sin0.1t)$ | A1 | 1.1b |
| | | (3) | |
| (d)(i) | $t=0, f=6 \Rightarrow A=6$ | M1 | 3.1b |
| | $t=0, r=20 \Rightarrow B=14$ | M1 | 3.3 |
| | $r=e^{0.3t}(20\cos0.1t+8\sin0.1t)=0$ | M1 | 3.1b |
| | $\tan0.1t=-2.5$ | A1 | 1.1b |
| | 2019 | A1 | 3.2a |
| (d)(ii) | 3750 foxes | B1 | 3.4 |
| (d)(iii) | E.g. <br> The model predicts a large number of foxes are on the island when the rabbits have died out and this may not be sensible. | B1 | 3.5a |
| | | (7) | |
| | | | (17 marks) |

| Notes |
|---|
| (a) |
| M1: Attempts to differentiate the first equation with respect to $t$ |
| M1: Proceeds to the printed answer by substituting into the second equation |
| A1*: Achieves the printed answer with no errors |
| (b) |
| M1: Uses the model to form and solve the auxiliary equation |
| A1: Correct values for $m$ |
| M1: Uses the model to form the CF |
| A1: Correct CF |
| (c) |
| M1: Differentiates the expression for the number of foxes |
| M1: Uses this result to find an expression for the number of rabbits |
| A1: Correct equation |
| (d)(i) |
| M1: Realises the need to use the initial conditions in the model for the number of foxes |
| M1: Realises the need to use the initial conditions in the model for the number of rabbits to find both unknown constants |
| M1: Obtains an expression for $r$ in terms of $t$ and sets = 0 |
| A1: Rearranges and obtains a correct value for tan |
| A1: Identifies the correct year |
| (d)(ii) |
| B1: Correct number of foxes |
| (d)(iii) |
| B1: Makes a suitable comment on the outcome of the model |

**Q5.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $(1+t)\frac{dP}{dt}+P=t^{\frac{1}{2}}(1+t)\Rightarrow\frac{dP}{dt}+\frac{P}{1+t}=t^{\frac{1}{2}}$ | B1 | 1.1b |
| | $I=e^{\int\frac{1}{1+t}dt}=1+t\Rightarrow P(1+t)=\int t^{\frac{1}{2}}(1+t)dt=...$ | M1 | 3.1b |
| | $P(1+t)=\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+c$ | A1 | 1.1b |
| | $t=0,\ P=5\Rightarrow c=5$ | M1 | 3.4 |
| | $P=\frac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}=\frac{\frac{2}{3}8^{\frac{3}{2}}+\frac{2}{5}8^{\frac{5}{2}}+5}{9}=...$ | M1 | 1.1b |
| | = 10 277 bacteria (allow awrt 10 300) | A1 | 2.2b |
| | | **(6)** | |
| (b) | $P=\frac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}\Rightarrow\frac{dP}{dt}=\frac{(1+t)\left(t^{\frac{1}{2}}+t^{\frac{3}{2}}\right)-\left(\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5\right)}{(1+t)^2}$<br>Alt: $P+(1+t)\frac{dP}{dt}=t^{\frac{1}{2}}+t^{\frac{3}{2}}\Rightarrow\frac{dP}{dt}=\frac{t^{\frac{1}{2}}+t^{\frac{3}{2}}-\frac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}}{(1+t)}$ | M1<br>A1ft | 3.4<br>1.1b |
| | $\left(\frac{dP}{dt}\right)_{t=1}=\frac{dP}{dt}=\frac{5\times10-\left(\frac{16}{3}+\frac{64}{5}+5\right)}{(5)^2}=\frac{403}{375}$ | dM1 | 3.1a |
| | $\frac{403}{375}\times1000=\frac{3224}{3}(=awrt1070)$ bacteria per hour | A1 | 3.2a |
| | | **(4)** | |

| | | | |
|---|---|---|---|
| | **(b) Alternative:** | | |
| | $P=\frac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}=\frac{\frac{16}{3}+\frac{64}{5}+5}{(1+4)}$ | M1 | 3.4 |
| | $=\frac{347}{75}$ | A1ft | 1.1b |
| | $(1+t)\frac{dP}{dt}+P=t^{\frac{1}{2}}(1+t)\Rightarrow5\frac{dP}{dt}+\frac{347}{75}=2\times5\Rightarrow\frac{dP}{dt}=\frac{403}{375}$ | dM1 | 3.1a |
| | $\frac{403}{375}\times1000=\frac{3224}{3}(=1075)$ bacteria per hour | A1 | 3.2a |
| | | **(4)** | |
| (c) | E.g.<br>• The number of bacteria increases indefinitely which is not realistic | B1 | 3.5b |
| | | **(1)** | |
| | | | **(11 marks)** |

| Notes |
|---|
| (a)<br>B1: A correct rearrangement (may be implied by subsequent work). Alternatively, recognises the LHS as a derivative and writes $(1+t)\frac{\mathrm{d}P}{\mathrm{d}t}+P=\frac{\mathrm{d}}{\mathrm{d}t}(P(1+t))\left(=t^{\frac{1}{2}}(1+t)\right)$ (may be implied).<br>M1: Uses the model to find the integrating factor (or recognise the derivative) and attempts the solution of the differential equation to achieve $P\times$ their IF $=\int$ their $t^{\frac{1}{2}}\times$ their IF $\mathrm{d}t=...$ but do not be too concerned with the mechanics of integrating the RHS but it must be attempted.<br>A1: Correct solution<br>M1: Interprets the initial conditions to find the constant of integration. Must be using $t=0$ and $P=5$ in an equation with a constant of integration, but their equation may have come from incorrect work. This is correctly interpreting the initial conditions and attempting to use them.<br>M1: Uses their solution to the problem to find the population after 8 hours. Must be using their solution, but allow for any equations which arise from an attempt at solving the differential equation.<br>A1: **cso** Correct number of bacteria (accept awrt 10 300) from a correct equation<br>(b)<br>M1: Realises the need to differentiate the model and uses an appropriate method to find the derivative. Allow the M for attempts at implicit differentiation with $(1+t)P=...$ Trivialised differentiation from incorrect work is M0.<br>A1ft: Correct differentiation of the correct answer to (a) up to the constant of integration to obtain $\mathrm{d}P/\mathrm{d}t$ in terms of $t$ (if implicit differentiation is used, they must get to a function in terms of $t$ only, or revert to the Alternative method). Follow through on their $c$ in an otherwise correct equation from (a).<br>M1: Uses $t=4$ in their $\mathrm{d}P/\mathrm{d}t$ (allow from any attempts at the derivative) to obtain a value for $\mathrm{d}P/\mathrm{d}t$.<br>A1: Correct answer, allow 1075 or answers rounding down to 1070 with correct units. Accept as 1.07 thousand bacteria per hour.<br>(NB If 5000 is used in (a) instead of 5, the answer here would be −198.725) |

| **Alternative:** |
|---|
| M1: Substitutes $t=4$ into their $P$<br>A1ft: Correct value for $P$. Follow through on their constant of integration from part (a), but the rest of the equation must be correct.<br>M1: Uses $t=4$ and their $P$ to find a value for $\mathrm{d}P/\mathrm{d}t$<br>A1: Correct answer allow 1075 or answers rounding down to 1070 with correct units. Accept as 1.07 thousand bacteria per hour.<br>(c)<br>B1: Suggests a suitable limitation which must refer to the model. Allow for a sensible comment even if they have no equation for the model<br>Do not allow answers such as "the model does not take account of external factors such as temperature" as we do not know what factors the model does take account of. |

**Q6.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | Pond contains 1000 + $5t$ litres after $t$ days | M1 | 3.3 |
| | If $x$ is the amount of pollutant in the pond after $t$ days<br>Rate of pollutant out = $20 \times \frac{x}{1000+5t}$ g per day | M1 | 3.3 |
| | Rate of pollutant in = 25 × 2 g = 50g per day | B1 | 2.2a |
| | $\frac{dx}{dt} = 50 - \frac{4x}{200+t}$ * | A1* | 1.1b |
| | | (4) | |
| (b) | $I = e^{\int \frac{4}{200+t} dt} = (200+t)^4 \Rightarrow x(200+t)^4 = \int 50(200+t)^4 \, dt$ | M1 | 3.1b |
| | $x(200+t)^4 = 10(200+t)^5 + c$ | A1 | 1.1b |
| | $x = 0, t = 0 \Rightarrow c = -3.2 \times 10^{12}$ | M1 | 3.4 |
| | $t = 8 \Rightarrow x = 10(200+8) - \frac{3.2 \times 10^{12}}{(200+8)^4}$ | M1 | 1.1b |
| | = 370g | A1 | 2.2b |
| | | (5) | |
| (c) | <u>Examples</u><br>• The model should take into account the fact that the pollutant does not dissolve throughout the pond upon entry<br>• The rate of leaking could be made to vary with the volume of water in the pond | B1 | 3.5c |
| | | (1) | |
| | | | **(10 marks)** |

| **Notes:** |
|---|
| (a)<br>**M1**: Forms an expression of the form 1000 + $kt$ for the volume of water in the pond at time $t$<br>**M1**: Expresses the amount of pollutant out in terms of $x$ and $t$<br>**B1**: Correct interpretation for pollutant entering the pond<br>**A1***: Puts all the components together to form the correct differential equation |
| (b)<br>**M1**: Uses the model to find the integrating factor and attempts solution of their differential equation<br>**A1**: Correct solution<br>**M1**: Interprets the initial conditions to find the constant of integration<br>**M1**: Uses their solution to the problem to find the amount of pollutant after 8 days<br>**A1**: Correct number of grams |
| (c)<br>**B1**: Suggests a suitable refinement to the model |

**Q7.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | The tank initially contains 100 L. 3 L are entering every minute and 2 L are leaving every minute so overall 1 L increase in volume each minute so the tank contains 100 + $t$ litres after $t$ minutes | M1 | 3.3 |
| | 2 L leave the tank each minute and if there are $S$g of salt in the tank, the concentration will be $\frac{S}{100+t}g/L$ so salt leaves the tank at a rate of $2\times\frac{S}{100+t}g$ per minute | M1 | 3.3 |
| | Salt enters the tank at a rate of $3\times 1g$ per minute | B1 | 2.2a |
| | $\therefore \frac{dS}{dt}=3-\frac{2S}{100+t}$ * cso | A1* | 1.1b |
| | | (4) | |
| (b) | $\frac{dS}{dt}+\frac{2S}{100+t}=3$ | | |
| | $I=e^{\int\frac{2}{100+t}dt}=(100+t)^2 \Rightarrow S(100+t)^2=\int 3(100+t)^2 dt$ | M1 | 3.1b |
| | $S(100+t)^2=(100+t)^3\ (+c)$<br>OR<br>$S(100+t)^2=30\,000t+300t^2+t^3\ (+c)$ | A1 | 1.1b |
| | $t=0,\ S=0\Rightarrow c=-10^6$ | M1 | 3.4 |
| | $t=10\Rightarrow S=100+10-\frac{10^6}{(100+10)^2}$ | dM1 | 1.1b |
| | OR<br>$S(100+10)^2=(100+10)^3\ (+c)\Rightarrow S=\ldots$ | | |
| | $=$ awrt 27 (g) or $\frac{3310}{121}$ (g) | A1 | 2.2b |
| | | (5) | |

| | | | |
|---|---|---|---|
| (c) | Concentration is $\left(100+t-\frac{10^6}{(100+t)^2}\right)\div(100+t)=0.9$ <br> OR <br> $S=0.9\ 100+t \Rightarrow 0.9\ 100+t\ =\ 100+t\ -\frac{10^6}{100+10\ ^2}$ <br> OR <br> $S=0.9\ 100+t\ \Rightarrow 0.9\ 100+t\ ^3=\ 100+t\ ^3-10^6$ | M1 | 3.4 |
| | $(100+t)^3=10^7 \Rightarrow t=\ldots$ <br> OR <br> $t^3+300t^2+30\,000t-9\,000\,000=0\Rightarrow t=\ldots$ | dM1 | 1.1b |
| | $t=$ awrt 115 (minutes) | A1 | 2.2b |
| | | (3) | |
| (d) | E.g. <br> • It is unlikely that mixing is instantaneous <br> • The model will only be valid when the tank is not full <br> • When the valve is closed, the model is not valid <br> • It is unlikely that the concentration of salt water entering the tank remains exactly the same | B1 | 3.5a |
| | | (1) | |
| | | | (13 marks) |

| Notes |
|---|
| (a) <br> M1: A suitable explanation for the "100 + $t$" e.g. as a minimum ($v$) = 100 + 3$t$ – 2$t$ = 100 + $t$ <br> M1: A suitable explanation for the $\frac{2S}{100+t}$ <br> There needs to be some explanation (words) for this part of the formula. <br> e.g. the concentration of (salt) $=\frac{S}{100+t}$ therefore (salt) out $=2\times\frac{S}{100+t}=\frac{2S}{100+t}$ <br> e.g. salt out $=\frac{2S}{\text{volume of water}}=\frac{2S}{100+t}$ <br> **Note:** M0 for $2\times\frac{S}{100+t}=\frac{2S}{100+t}$ only with no explanation <br> B1: Correct interpretation for the "3" e.g. salt in = 3 or $\frac{dS}{dt}$ in = 3 <br> <u>**Note:**</u> Salt water in = 3 is **B0** <br> A1*: Puts all the components together to form the given differential equation cso |

| |
|---|
| (b)<br>M1: Uses the model to find the integrating factor and attempts the solution of the differential equation. Look for $I.F. = e^{\int \frac{2}{100+t} dt} \Rightarrow S \times \text{'their } I.F.\text{'} = \int 3 \times \text{'their } I.F.\text{'} \, dt$<br>A1: Correct solution condone missing + $c$<br>**For the next three mark there must be a constant of integration**<br>M1: Interprets the initial conditions, $t = 0 \;\; S = 0,$ and uses in their equation to find the constant of integration.<br>dM1: Dependent on having a constant of integration. Uses their solution to the problem to find the amount of salt after 10 minutes.<br>A1: Awrt 27 or $\frac{3310}{121}$. (If the units are stated they must be correct)<br>**Note:** If achieves $S(100+t)^2 = 30\,000t + 300t^2 + t^3 + c$ the constant of integration $c = 0$ and the correct amount of salt can be achieved. If there is no + $c$ the maximum they can score is M1A1M0M0A0 |
| (c)<br>**Note:** Look out for setting $S = 0.9$ in this part, which scores no marks.<br>M1: Uses their solution to the model and divides by 100 + $t$ as an interpretation of the concentration and sets = 0.9.<br>Alternatively recognises that the amount of salt = 0.9(100 + $t$) and substitutes for $S$ in their solution to the model.<br>dM1: Dependent on previous method mark. Solves their equation to obtain a value for $t$. May use a calculator.<br>A1: Awrt 115 (If the units are stated they must be correct) or 1hr 45 mins with units<br>(d)<br>B1: Evaluates the model by making a suitable comment – see scheme for examples. |

**Q8.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a)(i) | Weight = mass × g $\Rightarrow m = \frac{30000}{g} = 3000$ <br> But mass is in thousands of kg, so $m = 3$ | M1 | 3.3 |
| (ii) | $\frac{dx}{dt} = 40\cos t + 20\sin t,\ \frac{d^2x}{dt^2} = -40\sin t + 20\cos t$ | M1 | 1.1b |
| | $3(-40\sin t + 20\cos t) + 4(40\cos t + 20\sin t)$ <br> $+ 40\sin t - 20\cos t = \ldots$ | M1 | 1.1b |
| | $= 200\cos t$ so PI is $x = 40\sin t - 20\cos t$ | A1* | 2.1 |
| | or | | |
| | Let $x = a\cos t + b\sin t$ <br> $\frac{dx}{dt} = -a\sin t + b\cos t,\ \frac{d^2x}{dt^2} = -a\cos t - b\sin t$ | M1 | 1.1b |
| | $4b - 2a = 200,\ -2b - 4a = 0 \Rightarrow a = \ldots, b = \ldots$ | M1 | 2.1 |
| | $x = 40\sin t - 20\cos t$ | A1* | 1.1b |
| (iii) | $3\lambda^2 + 4\lambda + 1 = 0 \Rightarrow \lambda = -1, -\frac{1}{3}$ | M1 | 1.1b |
| | $x = Ae^{-t} + Be^{-\frac{1}{3}t}$ | A1 | 1.1b |
| | $x = PI + CF$ | M1 | 1.1b |
| | $x = Ae^{-t} + Be^{-\frac{1}{3}t} + 40\sin t - 20\cos t$ | A1 | 1.1b |
| | | (8) | |
| (b) | $t = 0, x = 0 \Rightarrow A + B = 20$ | M1 | 3.4 |
| | $x = 0, \frac{dx}{dt} = -Ae^{-t} - \frac{1}{3}Be^{-\frac{1}{3}t} + 40\cos t + 20\sin t = 0$ <br> $\Rightarrow A + \frac{1}{3}B = 40$ | M1 | 3.4 |
| | $x = 50e^{-t} - 30e^{-\frac{1}{3}t} + 40\sin t - 20\cos t$ | A1 | 1.1b |
| | $t = 9 \Rightarrow x = 33\text{m}$ | A1 | 3.4 |
| | | (4) | |
| | | | **(12 marks)** |

| Notes: |
|---|
| **(a)(i)**<br>**M1**: Correct explanation that in the model, $m = 3$ |
| **(ii)**<br>**M1**: Differentiates the given PI twice<br>**M1**: Substitutes into the given differential equation<br>**A1***: Reaches $200\cos t$ and makes a conclusion<br>Or<br>**M1**: Uses the correct form for the PI and differentiates twice |

| |
|---|
| **M1**: Substitutes into the given differential equation and attempts to solve<br>**A1***: Correct PI<br>**(iii)**<br>**M1**: Uses the model to form and solve the auxiliary equation<br>**A1**: Correct complementary function<br>**M1**: Uses the correct notation for the general solution by combining PI and CF<br>**A1**: Correct General Solution for the model |
| **(b)**<br>**M1**: Uses the initial conditions of the model, $t = 0$ at $x = 0$, to form an equation in $A$ and $B$<br>**M1**: Uses $dx/dt = 0$ at $x = 0$ in the model to form an equation in $A$ and $B$<br>**A1**: Correct PS<br>**A1**: Obtains 33m using the assumptions made in the model. |

**Q9.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $100m^2+60m+13=0 \Rightarrow m=-0.3\pm 0.2\text{i}$ | M1 | 1.1b |
| | $x=\text{e}^{-0.3t}\left(A\cos 0.2t+B\sin 0.2t\right)$ | A1 | 1.1b |
| | PI: $x=2$ | B1 | 1.1b |
| | $x=\text{e}^{-0.3t}\left(A\cos 0.2t+B\sin 0.2t\right)+2$ | A1ft | 2.2a |
| | | **(4)** | |
| (b) | $t=0,\ x=0 \Rightarrow A=-2$ | M1 | 3.4 |
| | $\frac{\text{d}x}{\text{d}t}=-0.3\text{e}^{-0.3t}\left(-2\cos 0.2t+B\sin 0.2t\right)+\text{e}^{-0.3t}\left(0.4\sin 0.2t+0.2B\cos 0.2t\right)$<br>$t=0,\ \frac{\text{d}x}{\text{d}t}=10 \Rightarrow B=\ldots$(NB $B=47$) | M1 | 3.4 |
| | $x=\text{e}^{-0.3t}\left(47\sin 0.2t-2\cos 0.2t\right)+2$ | A1 | 1.1b |
| | $-0.3\text{e}^{-0.3t}\left(47\sin 0.2t-2\cos 0.2t\right)+\text{e}^{-0.3t}\left(9.4\cos 0.2t+0.4\sin 0.2t\right)=0$<br>$\Rightarrow t=\ldots$<br>or<br>$x=\sqrt{2213}\text{e}^{-0.3t}\sin(0.2t-0.0425)+2$<br>$\Rightarrow \frac{\text{d}x}{\text{d}t}=-0.3\sqrt{2213}\text{e}^{-0.3t}\sin(0.2t-0.0425)$<br>$+0.2\sqrt{2213}\text{e}^{-0.3t}\cos(0.2t-0.0425)$<br>$\Rightarrow t=\ldots$ | M1 | 3.1b |
| | $\tan 0.2t=\frac{100}{137}\Rightarrow 0.2t=0.630\ldots$<br>or<br>$\tan(0.2t-0.0425)=\frac{2}{3}\Rightarrow 0.2t=0.630$ | M1 | 2.1 |
| | $t=3.15\ldots$ weeks | A1 | 1.1b |
| | $x=\text{e}^{-0.3\times"3.15..."}\left(47\sin\left(0.2\times"3.15..."\right)-2\cos\left(0.2\times"3.15..."\right)\right)+2$ | M1 | 3.4 |
| | = awrt 12.1 {µg/ml} | A1 | 3.2a |
| | | **(8)** | |
| (c) | $t=10 \Rightarrow x=\text{e}^{-3}\left(47\sin(2)-2\cos(2)\right)+2=4.16\ldots$ | M1 | 3.4 |
| | The model suggests that it would be safe to give the second dose | A1ft | 2.2a |
| | | **(2)** | |
| | | | **(14 marks)** |

| Notes |
|---|
| (a)<br>M1: Uses the model to form and solve the auxiliary equation<br>A1: Correct CF, does not need $x =$<br>B1: Correct PI<br>A1ft: Deduces the correct GS (follow through their CF + PI). Must have $x = \text{f}(t)$ and PI not 0<br>(b)<br>M1: Uses the model and the initial conditions to establish the value of "$A$"<br>M1: Differentiates their model using the product rule and uses the initial conditions to establish the value of "$B$". Must be using $x = 0$ and $\frac{\text{d}x}{\text{d}t} = 10$<br>A1: Correct particular solution. This can be implied by the correct constants found following a correct answer to part (a).<br>M1: Uses their solution to the model with a correct strategy to obtain the required value of $t$ e.g. differentiates, sets equal to zero and solves for $t$<br>M1: Uses a correct trigonometric approach that leads to a value for $t$<br>A1: Correct value for $t$<br>M1: Uses the model and their value for $t$ to find the maximum concentration.<br>A1: Correct value<br>(c)<br>M1: Uses the model to find the concentration when $t = 10$<br>A1ft: Makes a suitable comment that is consistent with their calculated value<br>**Special case:** If the candidate's maximum value is less than 5 then<br>M1: never reaches 5 as maximum is…. or max is less than 5<br>A1: yes, it is safe |

**Q10.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $\frac{d^2x}{dt^2}=-5\frac{dx}{dt}+10\frac{dy}{dt}$ oe e.g. $\frac{dy}{dt}=\frac{1}{10}\left(\frac{d^2x}{dt^2}+5\frac{dx}{dt}\right)$ | B1 | 1.1b |
| | $\frac{d^2x}{dt^2}=-5\frac{dx}{dt}+10(-2x+3y-4)$<br>$=-5\frac{dx}{dt}-20x+\frac{30}{10}\left(\frac{dx}{dt}+5x+30\right)-40$<br>Or $\frac{1}{10}\left(\frac{d^2x}{dt^2}+5\frac{dx}{dt}\right)=-2x+\frac{3}{10}\left(30+5x+\frac{dx}{dt}\right)-4$ | M1 | 2.1 |
| | $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=50$* | A1* | 1.1b |
| | | (3) | |
| (b) | $m^2+2m+5=0\Rightarrow m=...$ | M1 | 3.4 |
| | $m=-1\pm 2\text{i}$ | A1 | 1.1b |
| | $m=\alpha\pm\beta\text{i}\Rightarrow x=e^{\alpha t}(A\cos\beta t+B\sin\beta t)=...$ | M1 | 3.4 |
| | $x=e^{-t}(A\cos 2t+B\sin 2t)$ | A1 | 1.1b |
| | PI: Try $x=k\Rightarrow 5k=50\Rightarrow k=10$ | M1 | 3.4 |
| | $GS: x=e^{-t}(A\cos 2t+B\sin 2t)+10$ | A1ft | 1.1b |
| | | (6) | |
| (c) | $\frac{dx}{dt}=e^{-t}(2B\cos 2t-2A\sin 2t)-e^{-t}(A\cos 2t+B\sin 2t)$ | B1ft | 1.1b |
| | $(y=)\frac{1}{10}\left(\frac{dx}{dt}+5x+30\right)=...$ | M1 | 3.4 |
| | $y=\frac{1}{10}e^{-t}((4A+2B)\cos 2t+(4B-2A)\sin 2t))+8$ | A1 | 1.1b |
| | | (3) | |

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (d) | $t=0, x=2\Rightarrow 2=A+10\Rightarrow A=-8$ | M1 | 3.1b |
| | $t=0, y=5\Rightarrow 5=\frac{1}{10}(2B-32)+8\Rightarrow B=1$ | M1 | 3.3 |
| | $x=e^{-t}(\sin 2t-8\cos 2t)+10$ | A1 | 2.2a |
| | $y=e^{-t}(2\sin 2t-3\cos 2t)+8$ | A1 | 2.2a |
| | | (4) | |
| (e) | E.g When $t>8$, the amount of compound $X$ and the amount of compound $Y$ remain (approximately) constant at 10 and 8 respectively, which suggests that the chemical reaction has stopped. This supports the scientist's claim. | B1 | 3.5a |
| | | (1) | |
| | | | (17 marks) |

| Notes |
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| (a)<br>B1: Differentiates the first equation with respect to $t$ correctly. May have rearranged to make $y$ the subject first. The dot notation for derivatives may be used.<br>M1: Uses the second equation to eliminate $y$ to achieve an equation in $x$, $\frac{dx}{dt}, \frac{d^2x}{dt^2}$.<br>A1*: Achieves the printed answer with no errors.<br>(b)<br>M1: Uses the model to form and attempts to attempts to solve the auxiliary equation (Accept a correct equation followed by two values for $m$ as an attempt to solve.)<br>A1: Correct roots of the AE<br>M1: Uses the model to form the complementary function. Must be in terms of $t$ only (not $x$)<br>A1: Correct CF<br>M1: Chooses the correct form of the PI according to the model and uses a complete method to find the PI<br>A1ft: Combines their CF (which need not be correct) with the correct PI to give $x$ in terms of $t$ so look for $x$ = their CF + 10<br>(c)<br>B1ft: Correct differentiation of their $x$. Follow through their $e^{\alpha t}(A\cos \beta t + B\sin \beta t)$<br>M1: Uses the model and their answer to part (b) to find an expression for $y$ in terms of $t$<br>A1: Correct equation. Mark the final answer but there is not need for terms to be gathered but must have $y = \ldots$. |

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| (d)<br>M1: Realises the need to use the initial conditions in the equation for $x$<br>M1: Realises the need to use the initial conditions in the equation for $y$ to find both unknown constants - must have equations from which both unknowns can be found. Alternatively, a complete method using $\frac{dx}{dt}$ to find the second constant is made.<br>A1: Deduces the correct equation for $x$<br>A1: Deduces the correct equation for $y$. For this equation constants should have been gathered.<br>(e)<br>B1: Allow for any appropriate comment with valid supporting reason. They must have equations of the correct form from (d). The coefficients may be incorrect, but they must have positive limits for each of $x$ and $y$.<br>Both $x$ and $y$ should be considered (see below for exception), and a reason and some comment about the suitability of the model made (though you may allow implicit conclusions). E.g.<br>• for values of $t > 8$, the amounts of compounds $X$ and $Y$ present settle at 10 and 8 without really varying, which supports the claim.<br>• $\frac{dx}{dt} \approx 0$ and $\frac{dy}{dt} \approx 0$ when $t = 8$, so neither are changing, which supports the claim.<br>• As $t$ gets large $x$ and $y$ tend to limits to 10 and 8 neither will be zero, hence the claim is not supported.<br>• $x = 10.0$ (awrt) and $y = 8.00$ (awrt) when $t = 8$, since neither is zero it is likely the reaction is still continuing so the claim is not supported.<br>Exception: Allow a reason that states the model assumes that the reaction continues indefinitely, so the claim is not supported. (The reaction stopping would require a change in the model.) |

Do NOT allow an answer that only considers $x$ or $y$. E.g. $x = 10$ when $t = 8$ so the model is not supported is B0 since there is no consideration that $y$ may be zero and hence end the reaction.

Alt for (c) and (d) restarting:

B1: Correct second order equation for $y$ formed: $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 40$

M1: Full method to obtain the general solution: they may recognise the similarity to the equation in $x$ and jump straight to finding the PI, or may form the aux equation etc again, but look for an attempt that combines a (correctly formed) CF and a PI. For this mark allow if the constants used are the same as those for the equation in $x$.

A1: Correct solution for $y$ with different constants than those for $x$, though allow recovery if they realise in (d) that they need different constants.

For (d)

M1: As main scheme, allow for using the initial conditions in one equation to make a start finding the constants.

M1: For a full method to obtain all four constants – if the same constant were used for both equations in (c) (inconsistently) then this mark cannot be scored. A full method here would, for instance, require finding $\frac{dx}{dt}$ and using this along with the given initial equations and initial conditions to find the second constants for each equation.

A1: One correct equation with SC of being qualified by the first M only if a full method to find both constants for just one equation is made (so M1M0A1A0 is possible in this case).

A1: Both equations correct.